DIVERGENCE THEOREM (GAUSS' THEOREM)

If S is a closed surface including region V in vector field $\underset{\sim}{F}$

$$\int_V div\,\underset{\sim}{F}\,dV = \int_S \underset{\sim}{F}\,.\,d\,\underset{\sim}{S}.$$

$$div\,\underset{\sim}{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

EXAMPLE:

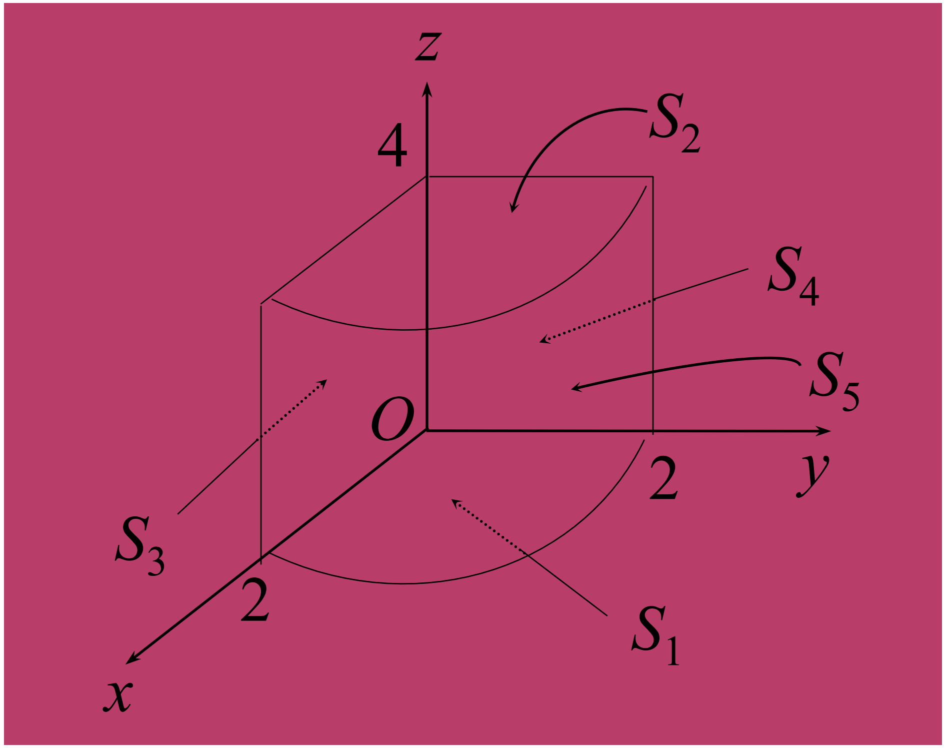
Prove Gauss' Theorem for vector field,

$\underset{\sim}{F} = x\underset{\sim}{i} + 2\underset{\sim}{j} + z^2\underset{\sim}{k}$ in the region bounded by

planes $z = 0, z = 4, x = 0, y = 0$ *and* $x^2 + y^2 = 4$

in the first octant.

Solution



For this problem, the region of integration is bounded by 5 planes :

$$S_1 : \ z = 0$$

$$S_2 : \ z = 4$$

$$S_3 : \ y = 0$$

$$S_4 : \ x = 0$$

$$S_5 : \ x^2 + y^2 = 4$$

To prove Gauss' Theorem, we evaluate both $\int_V div \underset{\sim}{F} \, dV$

and $\int_S \underset{\sim}{F} \, . \, d \underset{\sim}{S}$,

The answer should be the same.

1) We evaluate $\int_V div\,\underset{\sim}{F}\,dV$. Given $\underset{\sim}{F} = x\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + z^2\,\underset{\sim}{k}$.

So,

$$div\,\underset{\sim}{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(z^2)$$
$$= 1 + 2z.$$

Also, $\int_V div\,\underset{\sim}{F}\,dV = \int_V (1+2z)\,dV.$

The region is a part of the cylinder. So, we integrate by using polar coordinate of cylinder ,

$$x = \rho cos\phi\,; \quad y = \rho\sin\phi\,; \quad z = z$$
$$dV = \rho\,d\rho\,d\phi\,dz$$

where $0 \le \rho \le 2, 0 \le \phi \le \frac{\pi}{2}, 0 \le z \le 4.$

Therefore,

$$\int_V (1+2z)\,dV = \int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=0}^{2}\int_{z=0}^{4}(1+2z)\rho\,dz\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=0}^{2}\rho[z+z^2]_0^4\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=0}^{2}(20\rho)\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}}[10\rho^2]_0^2\,d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}}(40)\,d\phi$$

$$= [40\phi]_0^{\frac{\pi}{2}}$$

$$= 20\pi.$$

$$\therefore \int_V div\,\underset{\sim}{F}\,dV = 20\pi.$$

2) Now, we evaluate $\int_S \underset{\sim}{F} \,.\, d\underset{\sim}{S} = \int_S \underset{\sim}{F} \,.\, \underset{\sim}{n}\, dS.$

i) S_1 : $z = 0,\ \underset{\sim}{n} = -\underset{\sim}{k},\ dS = r\,dr\,d\theta$

$$\Rightarrow \underset{\sim}{F} = x\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + 0\,\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{F} \,.\, \underset{\sim}{n} = (x\,\underset{\sim}{i} + 2\,\underset{\sim}{j}).(-\underset{\sim}{k}) = 0$$

$$\therefore \int_{S_1} \underset{\sim}{F} \,.\, \underset{\sim}{n}\, dS = 0.$$

ii) S_2 : $z = 4, \underset{\sim}{n} = \underset{\sim}{k}, dS = r dr d\theta$

$$\Rightarrow \underset{\sim}{F} = x\underset{\sim}{i} + 2\underset{\sim}{j} + (4)^2\underset{\sim}{k} = x\underset{\sim}{i} + 2\underset{\sim}{j} + 16\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{F}.\underset{\sim}{n} = (x\underset{\sim}{i} + 2\underset{\sim}{j} + 16\underset{\sim}{k}).(\underset{\sim}{k}) = 16.$$

Therefore for S_2, $0 \le \text{r} \le 2$, $0 \le \theta \le \frac{\pi}{2}$

$$\therefore \quad \int_{S_2} \underset{\sim}{F}.\underset{\sim}{n}\, dS = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} 16\, r dr d\theta$$

$$= \cdots$$

$$= 16\pi.$$

iii) S_3 : $y = 0,\ \underset{\sim}{n} = -\underset{\sim}{j},\ dS = dxdz$

$$\Rightarrow \underset{\sim}{F} = x\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + z^2\,\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{F} \cdot \underset{\sim}{n} = (x\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + z^2\,\underset{\sim}{k}).(-\underset{\sim}{j})$$

$$= -2.$$

Therefore for S_3, $0 \le x \le 2,\ \ 0 \le z \le 4$

$$\therefore \quad \int_{S_3} \underset{\sim}{F} \cdot \underset{\sim}{n}\, dS = \int_{x=0}^{2} \int_{z=0}^{4} (-2)\, dz dx$$

$$= \cdots$$

$$= -16.$$

iv) S_4 : $\quad x = 0,\ \underset{\sim}{n} = -\underset{\sim}{i},\ dS = dydz$

$$\Rightarrow \underset{\sim}{F} = 0\underset{\sim}{i} + 2\underset{\sim}{j} + z^2\underset{\sim}{k} = 2\underset{\sim}{j} + z^2\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{F} \cdot \underset{\sim}{n} = (2\underset{\sim}{j} + z^2\underset{\sim}{k}).(-\underset{\sim}{i}) = 0.$$

$$\therefore \quad \int_{S_4} \underset{\sim}{F} \cdot \underset{\sim}{n}\, dS = 0.$$

v) $S_5: \quad x^2 + y^2 = 4,\ dS = \rho\, d\phi\, dz$

$$\nabla S_5 = 2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j} \quad and \quad \left|\nabla S_5\right| = 4$$

$$\Rightarrow \underset{\sim}{n} = \frac{\nabla S_5}{\left|\nabla S_5\right|} = \frac{2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j}}{4}$$

$$= \frac{1}{2}(x\,\underset{\sim}{i} + y\,\underset{\sim}{j}).$$

By using polar coordinate of cylinder :

$$x = \rho\cos\phi, y = \rho\sin\phi, z = z$$

where for S_5 :

$$\rho = 2,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le z \le 4,\ dS = 2\,d\phi\, dz$$

$$\Rightarrow \underset{\sim}{F} \,.\, \underset{\sim}{n} = (x\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + z^2\,\underset{\sim}{k}).\left(\frac{1}{2}x\,\underset{\sim}{i} + \frac{1}{2}y\,\underset{\sim}{j}\right)$$

$$= \frac{1}{2}x^2 + y$$

$$= \frac{1}{2}(\rho\cos\phi)^2 + (\rho\sin\phi)$$

$$= 2\cos^2\phi + 2\sin\phi; \qquad \text{kerana } \rho = 2.$$

$$= 2(\cos^2\phi + \sin\phi).$$

$$\therefore \int_{S_5} \underset{\sim}{F} \,.\, \underset{\sim}{n}\, dS = \int_{\phi=0}^{\frac{\pi}{2}}\int_{z=0}^{4} (2)(\cos^2\phi + \sin\phi)(2)\, d\phi dz$$

$$= \cdots$$

$$= 16 + 4\pi.$$

Finally,

$$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_{S_3} \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_{S_4} \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_{S_5} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$= 0 + 16\pi - 16 + 0 + 16 + 4\pi$$

$$= 20\pi.$$

$$\therefore \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = 20\pi.$$

$LHS = RHS$

$\Rightarrow$ Gauss' Theorem has been proved.